APPLICATION OF THE THEORY OF SYMMETRY GROUPS TO THE CATALOGUING OF SQUARE CERAMIC TILINGS.

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SUMMARY

This paper has been motivated by the lack of a system of classification for tilings composed of square tiles, which is simple to use in specific applications of ceramic coverings. The paper uses the theory of symmetry groups on the plane as a basis for a primary classification of tilings made up of square tiles, rectangular tiles or sets of tiles which form a square or rectangular surface.

In parallel to the theoretical development, this paper shows existing tilings to illustrate some of the tilings described, with a table of occurrence of the different types of square tiles and the different tilings.

The result obtained is a large number of tilings which may theoretically be generated with the use of a single type of tile. This result suggests the obtention of an infinite number of tilings provided consideration is made not of a single type of tile but rather a set of tiles which form a square or rectangular surface.

1.- INTRODUCTION

The different treatments given to the subject of tilings and mosaics from such diverse viewpoints as the historical, geometrical, artistic, architectural, and crystallographic aspects or the algorithms used in computer graphics have inspired the basis of this paper, whose objective is the cataloguing of tilings.

This objective is justified by the fact that application of the theory of groups is not synthesized for the cataloguing of ceramic coverings from the point of view of the multiplicity of tilings which may be generated from a single type of tile.

The different ways in which the various tilings may be generated is a result of the application of the theory of symmetry groups on the plane which has been fully covered in the past. For this reason the procedure followed to carry out this work is based on the compilation of the definitions of the different symmetry groups on the plane also known as crystallographic groups on the plane or periodic groups, carried out by: Coxeter (1984), Rose and Stanford (quoted in Alsina and Trillas, 1984), and Grünbaum and Shephard (1987), although we recognise that there exist others formulated by other researchers.

The application of these plane symmetry groups to a marked square tile (where a square tile is understood to be that which bears a motif on its surface which assigns a certain point symmetry group to the tile) is the point of departure of this study thus obtaining a primary classification of the tilings formed by this type of tile.

From this classification we have sought a correlation with the classifications carried out by Grünbaum and Shephard (1987).

2. BASIC CONCEPTS

Transformation on the plane

A transformation T on a set σ (of geometrical entities or transformations) is an action which changes the initial state of σ to a final state of σ'. In it a one-to-one correspondence is established between all the points on the plane. The action of transformation may be denoted as $T\sigma = \sigma'$. If P is another transformation, then $PT\sigma$ is understood to be $P(T\sigma)$.

Isometry

An isometry is a transformation on the plane which conserves the distance, in such a way that if p, q, are any two points on the plane to which an isometry is applied, transforming them into p´, q´, segments pq and p´q´ shall be equal. Two types of isometry may occur on the plane.

Direct isometry, which conserves the direction and which may be:
- a rotation around a point called the centre of rotation.
- a translation in any direction.

Opposed isometry, which inverts the direction, and which may be:
- a reflection in a line.
- A glide-reflection, reflection in a line combined with a translation parallel to the line.

Symmetry

A symmetry is an isometric transformation which produces an image state indistinguishable from the initial state. If S is a symmetry of σ, then $S\sigma = \sigma$.

Symmetry Group.

$S(\sigma)$ is a symmetry group of a set σ , if it contains all the symmetries of σ, i.e. if it contains all the isometric transformations which produces an image state indistinguishable from the initial state. The transformations of $S(\sigma)$ form a group if they satisfy the following conditions:

1.- given any two elements A,B of $S(\sigma)$, their product AB is in $S(\sigma)$.

2.- given any three elements A,B,C of $S(\sigma)$, $A(BC)=(AB)C$.

3.- there exists an element I in $S(\sigma)$, called the identity element such that $IA=A$ for each element A of $S(\sigma)$.

4.- given any element A of $S(\sigma)$, there exists an element A^{-1} in $S(\sigma)$ called the inverse of A, such that

$$A A^{-1} = A^{-1} A = I.$$

The number of mutually different transformations is called the "order" of the group. The set has symmetry or is symmetric if the order of $S(\sigma)$ is greater than or equal to 2. The set is asymmetrical if the order of $S(\sigma)$ is equal to 1, i.e. if $S(\sigma)$ contains only the identity element.

Point Symmetry Groups.

These may be cyclical and dihedral:

a) Cyclical groups of the order n (Cn) are symmetry groups which contain "n" angle rotations "$2\pi /n$" around a fixed point called the centre of rotation. If n=1, a cyclical group C1 exists which contains a single transformation, the identity.

b) Dihedral groups of the order 2n (Dn) are symmetry groups which contain "n" rotations around a fixed point and "n" reflections in "n" lines equally inclined to each other which pass through the same fixed point. If n=1 (D1 of the order 2) there exists the dihedral group which contains the identity and a reflection in a line. If n=2 (D2 of the order 4), there exists the dihedral group which contains the identity, two perpendicular reflections and an angle "π ", around a fixed point in which the lines of reflection intersect. When "n" is greater than or equal to 3, Dn is the symmetry group of a regular polygon with "n" sides. Finally, if $n=\infty$, $D\infty$ contains all the rotations around a fixed point and all the reflections of lines passing through said point. This is the symmetry group of a circular disc.

All groups Cn, Dn ("n" greater or equal to 1), have the property of possessing a fixed plane point, and are therefore called point symmetry groups, and are the only symmetry groups which may have a closed compact set.

Symmetry Groups in a Ribbon.

If the symmetry group contains a translation {na}, where "n" is an integer and "a" a vector of direction, where any other isometry may be contained, we have an infinite one-dimensional group, or symmetry group of a ribbon, strip or frieze.

Symmetry Groups on the Plane

If the symmetry group contains two independent translations whose direction is neither parallel nor opposed {na+mb} where "n"and "m" are integers and "a" and "b" are fixed vectors, where any other isometry may be contained, we have an infinite two-dimensional group or symmetry group on the plane.

If we take a fixed point on the plane and apply to it all the translations obtained by combining "n" translations of "a" and "m" translations of "b", we shall obtain a lattice of points. In general, this lattice of points may be seen as the vertices of a parallelogram. The vectors "a" and "b" correspond to any pair of adjacent sides of a parallelogram in the lattice. From this lattice may be extracted a region understood to be the minimum region on the plane which, by means of the application of the appropriate isometries will define the composition of the parallelogram.

Crystallographic Constraint.

W. Barlow (quoted in Alsina and Trillas, 1984) demonstrated that the only rotations which may form part of a symmetry group are those of the order 2, 3, 4 or 6, i.e. the rotations of an angle of 180°, 120°, 90° and 60°. With this constraint, there only exist 7 symmetry groups in ribbons and 17 symmetry groups on the plane.

This paper is centred solely on the generation of the 17 symmetry groups on the plane. In order to distinguish each of these we have used the international group symbol (Grünbaum and Shephard, 1987, p.44).

Definition of the transformations which make up each of the 17 symmetry groups on the plane varies according to the researcher. Table 1 shows those used by Coxeter (1984), Rose and Stafford (quoted in Alsina and Trillas, 1984) and Grünbaum and Shephard (1987).

Diagram of the Symmetry Group.

By diagram of the symmetry group (Grünbaum and Shephard, 1987, p.37), we understand that group in which the transformations, (reflections, glide-reflections, rotations and translations) which define the symmetry group have been represented.

Isomorphism

Two symmetry groups $S(\sigma1)$ and $S(\sigma2)$ are isomorphic, if the group diagram of one can be made to coincide with the group diagram of the other by application of a suitable affinity (Grünbaum and Shephard, 1987, p.38). Geometrically, this means that any lattice may be made to coincide with another, using a suitable affinity. This may be described as the application of a rigid movement, a change of scale or a change in the angle between axes. If $S(\sigma1)$ and $S(\sigma2)$ are isomorphic, then $\sigma1$ and $\sigma2$ have the same point of symmetry.

Flat Tiling

This is a countable family of compact closed sets which cover the plane without gaps or overlaps. The different closed compact sets are called tiles. The theory of symmetry groups may be applied to the tilings, thus to arrive at the following concepts:

Symmetry Groups of a Tile S(B): this is the set of symmetries which produce an image state of the tile which is indistinguishable from the initial one.

Symmetry Group of a Tiling S(E): this is the set of symmetries which trace each tile in the tiling onto any other.

3.- CONSTRAINTS OF THE PROBLEM

Classification of tilings by application of the group theory must face a great variety of factors which may be involved therein, from the different tile shapes which may exist to the relative arrangement of each tile with respect to the adjacent tiles, or the number of different tiles which may appear in a tiling, as well as with many other factors. For this reason we have introduced the following constraints in this first phase:

1.- All the tiles which make up the tiling are equal, in shape and size.

2.- The tiling is edge to edge: i.e. the corners of the tiles coincide with the vertices of the tiling, and the sides of the tile with the edges of the tiling, and each side of a tile is precisely the side of another tile.

3.- The tiling is periodic and symmetrical, i.e. there exist in the symmetry group of the tiling S(E) at least two non-parallel translations and any symmetry in addition to the identity.

4.- All the symmetries of the tile are symmetries of the tiling, i.e. the transformations which constitute S(B) are also transformations which are in S(E). This is analogous to considering that for any two tiles in tiling E, there exists a symmetry in S(E) which traces one of the tiles onto the other, and that for each tile, the symmetries of S(B) are in one-to-one correspondence with the symmetries of S(E) which trace B on itself.

This constraint is equivalent to considering only isohedral tilings. Such tilings have been defined by Grünbaum and Shephard (1987, p.31) as follows: "if all the tiles in a tiling E are equivalent, i.e. if the tiling's symmetry group S(E) contains a transformation which traces one tile onto another, the tiles are said to form a transitive class and the tiling is said to be tile-transitive or isohedral.

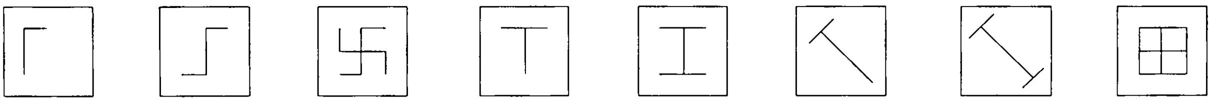
5.- The shape of the tile making up the tiling is square, leading us to a symmetry group for the tile of D4 (dihedral group of order 8). The symmetries of a square which leave it indistinguishable from the initial one are:

- the identity
- the 4 reflections in the lines which, passing through the centre of the tile, form 45°, 90°, 135°,180°.
- the 3 rotations in the centre of the tile of 90°, 180°, 270°.

At the same time, the square tile generates a tiling E, whose symmetry group S(E) is P4M. This means that the symmetry groups which contain rotations of 60° and/or 120°, since they are not subgroups of D4 or P4M, may not be generated by a square tile. This eliminates the following symmetry groups in the tile: C3, C6, D3, D6, and the following symmetry groups in the plane: P3, P3M1, P31M, P6, P6M.

Finally, if we consider that the tile bears a marking on its surface, the symmetry group of the former shall be modified by inclusion of the latter. In this case, the symmetry groups of the marked tile S(Bm) and the symmetry group of the marked tiling S(Em), are necessarily subgroups of the symmetry groups corresponding to the basic unmarked tiling. In other words, the symmetry groups S(B) and S(E) of the unmarked tiling will tend to be reduced when the mark is introduced in the tile, given that isometries may exist which are symmetries of the unmarked tiling E but which are not symmetries of the marked tiling Em (Grünbaum and Shephard, 1987, p.270).

The possible marks or patterns which may be borne by a tile are infinite if consider the design in itself (vegetable, geometrical, heraldic, etc.). Nonetheless, we may analyze the symmetries of the pattern to obtain eight different square marked tiles which differ in their symmetry groups. For all this we have used as the basic pattern an "L" shape, like that used by Grünbaum and Shephard (1987, p.270), from whence we have also assumed the distinction of the symmetry groups in the tile, D1 and D2 according to whether the symmetries refer to the axes which pass through the mid-points of the sides or to the axes determined by the diagonals of the square, using D1C or D2C when they refer to the axes which pass through the mid-points of the sides, and D1L or D2L when they correspond with the axes determined by the diagonals.



S(B)=C1 S(B)=C2 S(B)=C4 S(B)=D1C S(B)=D2C S(B)=D1L S(B)=D2L S(B)=D4

- FIGURE 1 -

4.- ANALYSIS OF THE SYMMETRY GROUPS IN THE PLANE

In order to carry out this analysis, the diagrams of the symmetry groups in the plane have been represented, i.e. the rotations, translations, reflections and glide-reflections which define or generate it have been traced, together with the arrangement of the pattern corresponding to each group (Alsina y Trillas, 1984).

In each diagram we have looked for the possible fundamental parallelograms which allow the plane to be covered by means of two parallel translations to the sides thereof.

We have then looked for the minimum region made up by each fundamental parallelogram, in such a way that, by means of the application to this region of isometries contained in the symmetry group the tiling is generated.

In this way we have obtained the minimum regions which, by means of the application of certain isometries, are capable of generating a tiling of parallelograms. This tiling of parallelograms is directly related with the specific symmetry group of the plane, since it has been obtained from it.

Once this analysis has been carried out, we have assimilated the minimum region to one tile, on the basis of which we have made the following considerations:

1.- If two tilings have the same type of symmetry, their group diagrams are isomorphic, and we may, by means of the application of a suitable affinity (rigid movement, change of scale or change of angle between axes), make not only the group diagram, but also the minimum regions, coincide. For this reason the minimum region, which always has the shape of a parallelogram, may be made to coincide with a square. Furthermore, considering the concept of isomorphism, we may extend the subject of tiling classification and consider the classification of tilings made up of rectangular tiles.

This extension of the classification only requires that it be considered that the rectangle corresponds to a symmetry group D2 (dihedral group of Order 4) and this rectangular tile generates a tiling E whose symmetry group S(E) is PMM.

If we consider that the tile bears a pattern on its surface, the symmetry group of the former shall be modified by the latter. Analogously to that stated above for the square marked tile the rectangular marked tile may only be of four different types which differ in their symmetry group. Since the symmetry groups of the rectangular marked tile must be sub-groups of the unmarked tile, these shall be: C1, D1, C2, D2.

2.- The symmetry groups of the tiles with cyclical sets, C1, C2, C3 and C4 do not contain reflections, for which reason these tiles cannot generate tilings with symmetry groups which contain reflections. This constraint places an excessive restriction on the classification which we are seeking, and therefore if we allow that a tile with a cyclical symmetry group may generate a tiling with a symmetry group which contains reflections, we are allowing a tiling made up of two types of tiles, the original one and the symmetrical one. This leads us to consider the possibility not only of making a classification for the case of tilings made up of a single type of tile, but also of attempting to classify at the same time tilings made up of sets of tiles.

A set of tiles, following the criteria used for one tile, has been assimilated into the minimum region, for which reason the set of tiles must necessarily define a square or rectangular surface. Likewise, this set of tiles must have an associated point symmetry group, which shall be either one of those corresponding to the square marked tile, or those of the rectangular marked tile.

3.- Certain isometries applied to the minimum region necessarily require to be applied on minimum regions of a square shape. For this reason we have indicated in the drawings two types of shapes for the minimum region or tile: square and rectangular, so that the appearance of a square shape indicates that the tile or sets of tiles must necessarily be square in shape and the appearance of a rectangular shape indicates that the tile or set of tiles may be either square or rectangular.

Figures 2 and 7 show the diagrams of the 12 symmetry groups in the plane compatible with the square tile. In each is indicated:

- the isometries which characterize it.
- the fundamental parallelograms which may be defined.
- the minimum regions which may be extracted from each fundamental parallelogram.
- the different square marked tiles, rectangular marked tiles or sets of marked tiles which can generate a tiling with that determined symmetry group, with an indication in each one of the isometries which must be applied to them.

5.- RESULTS.

As a consequence of the above analysis, we can make a primary classification of tilings made up of square marked tiles, rectangular marked tiles or sets of marked tiles which form a square or rectangular surface, considering the symmetry group of the tile and the isometries applied thereto (rotation, translation, reflection, glide-reflection). For this purpose we group together all tiles of the same type, i.e. those with the same symmetry group, indicating the isometries applied to each of them which lead us to a determined symmetry group in the tiling.

The nomenclature of any tiling must indicate the symmetry group of the tile or set of tiles and the symmetry group of the tiling in that order.

The distinction between the tiling made up of a single type of tile, square or rectangular, and the tiling made up of a set of tiles, is made by means of the following nomenclature procedure:

a) Tiling made up of a single type of square tile: S(B) - S(E)

b) Tiling made up of a single type of square tile: S(B),R - S(E)

c) Tiling made up of a set of tiles: S(B) (x,y) - S(E)

where "x" is the number of tiles on one side and "y" the number of tiles on the adjoining side.

If the set of tiles forms a square surface, we have abbreviated it as follows: S(B) (x) - S(E)

In the case where there exists for the same type of tiles various tilings with the same symmetry group, these are distinguished by means of the addition of a number in parentheses:

S(B) - S(E)(1)
S(B) R - S(E)(1)
S(B) (x,y) - S(E)(1)

Tables 2 and 3 show the proposed classification. In these, starting from the symmetry group of the tile or set of tiles, and considering the isometries applied to each tile, one obtains the nomenclature of the tiling which we propose. These tables also show the correspondence between these nomenclatures and those proposed by Grünbaum and Shephard (1987).

6.- CONCLUSIONS.

The classification obtained makes it possible to catalogue edge-to-edge tilings made up of square tiles, rectangular tiles or sets of tiles which form a square or rectangular surface, according to the symmetry group of the tile forming the tiling and according to the symmetry group of the tiling. The latter is derived from the application of determined isometries to the tile or set of tiles.

On analysis of the different diagrams of the symmetry groups in the plane, the fundamental parallelograms, we have observed that:

1-. If the arrangement of the patterns in the fundamental parallelogram responds to or is a consequence of the isometries which define the symmetry group of said parallelogram, the latter determines a tile with a certain symmetry group which leads us to a symmetry group of the tiling, in accordance with the classification proposed. When this occurs, the fundamental parallelogram is one of the minimum regions and the tiling is isohedral.

2.- If the arrangement of the patterns in the fundamental parallelogram is the consequence of the addition to the isometries which make up the symmetry group of said parallelogram, of other isometries not included in the symmetry group thereof, this parallelogram determines a tile with a certain symmetry group which leads us to a symmetry group on the tiling which shall be different to that indicated in the classification. When this occurs, the parallelogram shall not be a minimum region and the tiling is not isohedral. These cases are shown in the tables of diagrams of symmetry groups in the plane using an asterisk in parentheses (*), and its classification is left to future studies.

It is also interesting to indicate that in the case of the tiling with symmetry group C1, we have found that the application of the isometries corresponding to a determined symmetry group in the plane may generate visually different tilings. This occurs when the arrangement of the isometries which are to be applied to the tile may be located in different positions, i.e. when keeping the marked tile with the same orientation, the positions of the isometries which are to be applied to it are moved cyclically in a single direction. In this way the relative positions between the pattern and the isometries are modified, this generating visually different tilings. These cases are shown in Figures 2 to 7 by means of a number in parentheses which indicates the number of visually different tilings which may be produced.

It has been considered illustrative to give examples of the only tilings which may be generated with square tiles. Figure 8 shows the tilings which may be obtained using a tile with cyclical group C1, C2 or C4, and Figure 9 shows those obtained with dihedral tiles D1C, D1L, D2C, D2L o D4.

From these examples we may deduce that the tile C1 is that which may generate the greatest number of tilings (6 different ones), because the symmetry group of the tile only contains the identity (Order 1), and the tiles which can only generate one tiling are those corresponding to symmetry group C4 (Order 4) and symmetry group D4 (Order 8). In other words, as the order of the symmetry group of the tile is increased, the possibility of obtaining different tilings made up of a single type of tile decreases.

The search for real examples which validate the practical application of this work is shown in Table 4. These examples, obtained from Barnard (1979), Barros and Almasque (1988), Catleugh (1983), Gonzalez (1952), Lemmen (1988) and Porcar (1987), indicate that most of the applications correspond to tiles with symmetry group D1L (48 out of 121) or D4 (31 out of 121), which generate tilings with symmetry group P4M (84 out of 121). The other possible tilings are used in very isolated applications in specific cases of ceramic coverings. With the same object, Figures 10 and 11 show two fragments of pavements from the Museum of Decorative Arts in Prague which exemplify the D2L-P4M type of tiling according to the proposed classification, while Figure 12 shows another fragment as an example of Tiling D1C(2,1)-PMG.

Finally some aspects are set out in which the authors are currently working, within this line of research.

- modification of the square or rectangular shape by shapes which are also four-sided, but with angles in the vertices which are not 90º.

- modification of the square or rectangular shape by shapes which have a number of sides other than four (triangle, pentagon, hexagon, etc.).

- modification of the set of square tiles which form a square or rectangular surface, by a set of square tiles which do not form a rectangular surface.

- use of combinations of sets of tiles of different shapes.

- introduction of the pattern drawing (geometrical, heraldic, anthropomorphic, floral, vegetable, etc.) into the classification.

- introduction of colour into the classification.

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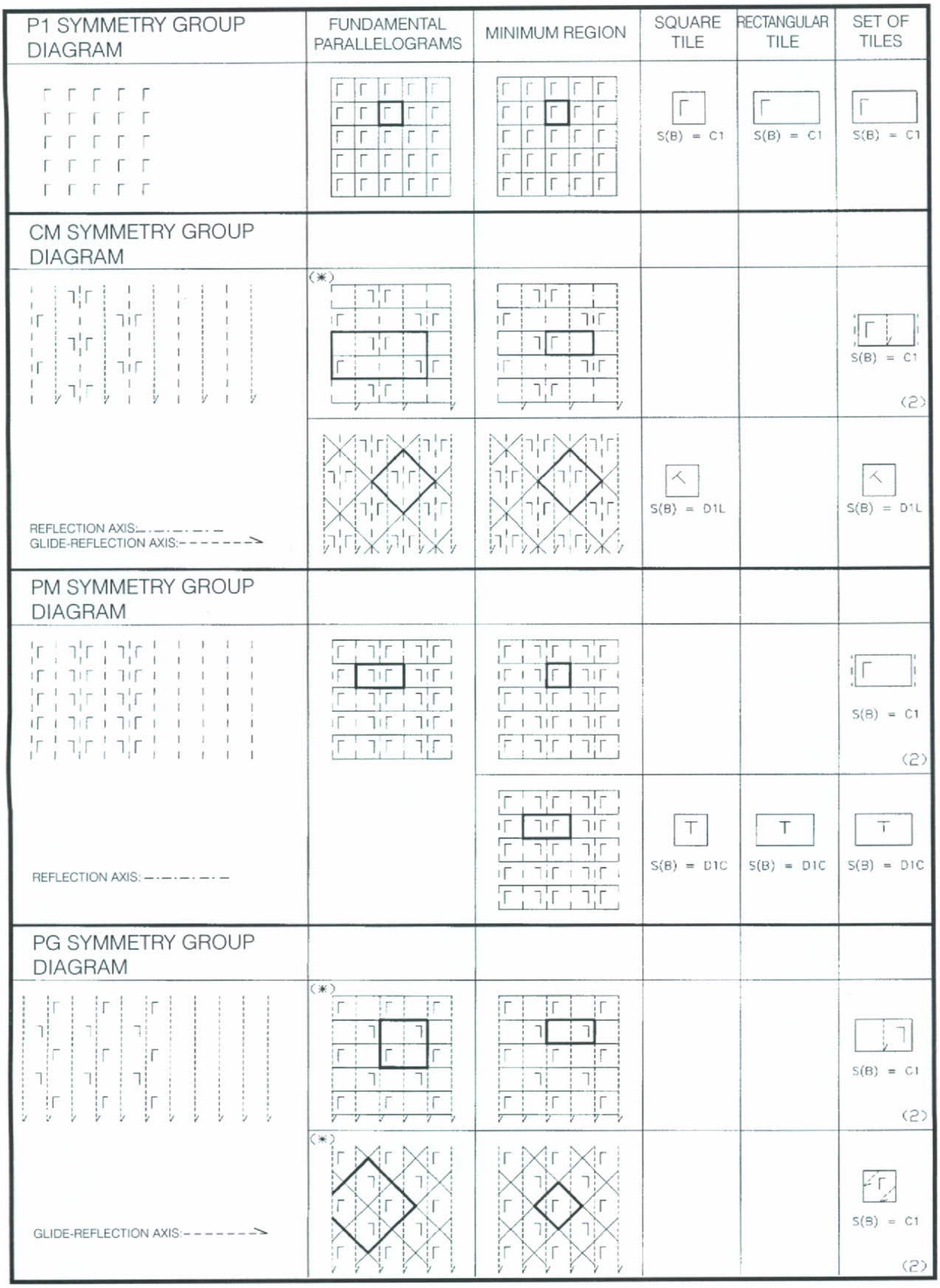
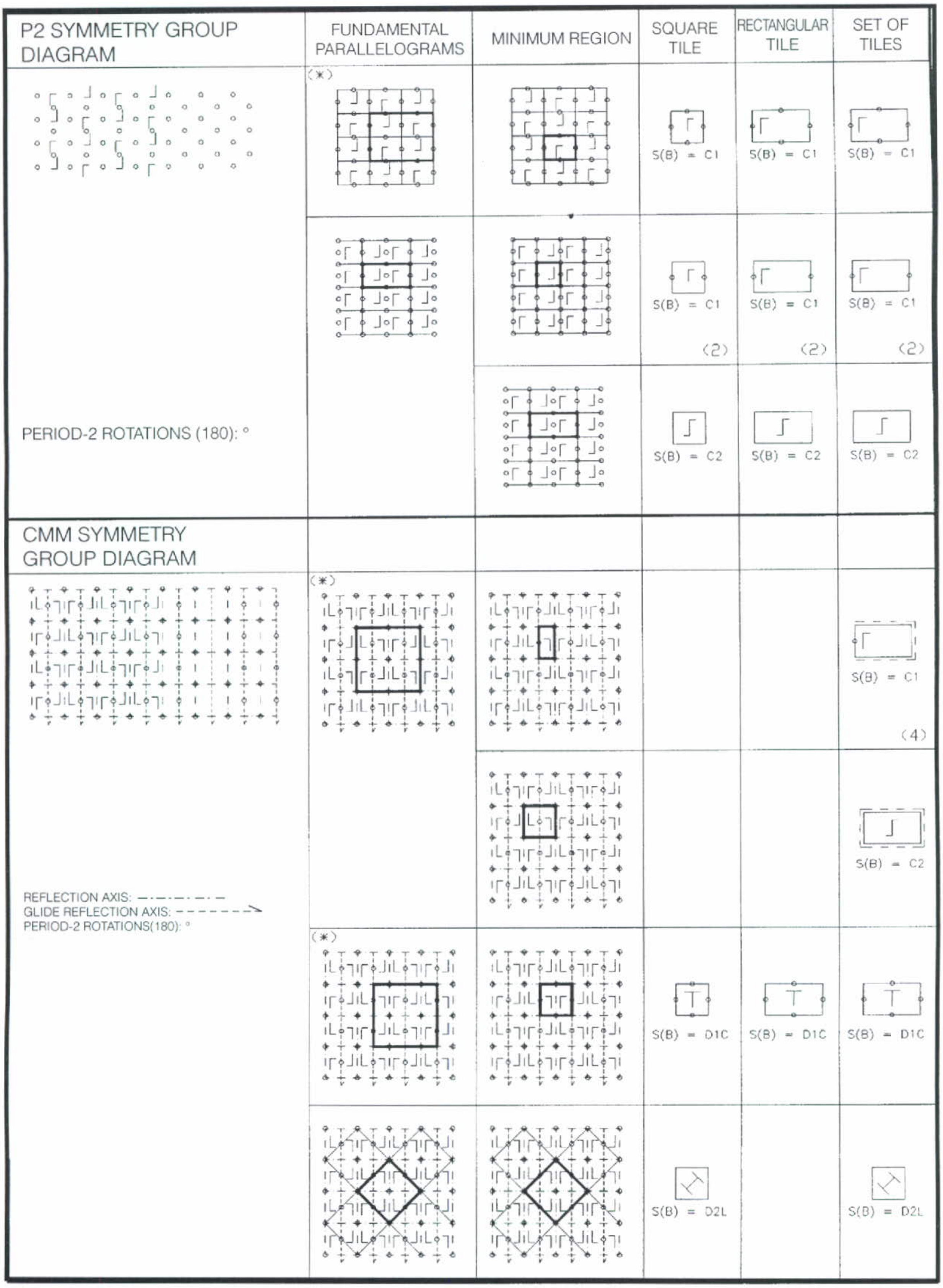
International Group Symbol	Coxeter	Rose and Stafford	Grünbaum and Shephard
P1	2 translations	No symmetry axes No glide-reflection axes Maximum turning order = 1	2 translations
PM	2 reflections	Symmetry axes No perpendicular symmetry axes No glide-reflection axes	2 translations 2 reflections
PG	2 parallel glide-reflections	Glide-reflection axes No symmetry axes No perpendicular glide-reflection axes	2 translations 2 glide-reflections
CM	1 reflection 1 parallel glide-reflection	Symmetry axes Glide-reflection axes parallel to those of symmetry No perpendicular symmetry axes No glide-reflection axes perpendicular to those of symmetry	2 translations 1 glide-reflection 1 reflection
P2	3 half-turns	No symmetry axes No glide-reflection axes Maximum turning order = 2	2 translations 4 Period 2 rotations
CMM	2 perpendicular reflections 1 half-turn	Symmetry axes Perpendicular symmetry axes No Order 6 or Order 3 turning centre Symmetry group of the rectangle formed by symmetry axes = C2	2 translations 2 glide-reflections 2 reflections 4 Period 2 rotations
PMM	The reflections in the four sides of a rectangle	Symmetry axes Perpendicular symmetry axes No Order 6 or Order 3 turning centre Symmetry group of the rectangle formed by symmetry axes = C1	2 translations 4 reflections 4 Period 2 rotations
PGG	2 perpendicular glide-reflections	Reflection axes Perpendicular glide-reflection axes No symmetry axes	2 translations 2 glide-reflections 2 Period 2 rotations
PMG	1 reflection 2 half-turns	Symmetry axes Glide-reflection axes perpendicular to those of symmetry No perpendicular symmetry axes	2 translations 2 glide-reflections 1 reflection 2 Period 2 rotations
P3	2 120° turns	No symmetry axes No glide-reflection axes Maximum turning order = 3	2 translations 3 Period 3 rotations
P3M1	1 reflection 1 120° turn	Symmetry axes Perpendicular symmetry axes Order 3 turning centre Symmetry group of the triangle formed by three symmetry axes such that no other symmetry axis intersects it = C1 No Order 6 turning centre	2 translations 1 glide-reflection 1 reflection 3 Period 3 rotations
P31M	The reflections in the three sides of an equilateral triangle	Symmetry axes Perpendicular symmetry axes Order 3 turning centre Symmetry group of the triangle formed by three symmetry axes such that no other symmetry axis intersects it = C3 or D3 No Order 6 turning centre	2 translations 1 glide-reflection 1 reflection 2 Period 3 rotations
P4	1 half-turn 1/4 turn	No symmetry axes No glide-reflection axes Maximum turning order = 4	2 translations 1 Period 2 rotation 2 Period 4 rotations
P4G	1 reflection 1/4 turn	Symmetry axes Perpendicular symmetry axes Symmetry group of the rectangle formed by symmetry axes = C4 No Order 6 or Order 3 turning centres	2 translations 2 glide-reflections 1 reflection 1 Period 2 rotation 1 Period 4 rotation
P4M	The reflections in the three sides of a triangle with angles of 45°, 45° and 90°	Symmetry axes Perpendicular symmetry axes Symmetry group of the rectangle formed by symmetry axes = D1 or D2 No Order 6 or Order 3 turning centres	2 translations 1 glide-reflection 3 reflections 1 Period 2 rotation 2 Period 4 rotations
P6	1 half-turn 1 120° turn	No symmetry axes No glide-reflection axes Maximum turning order = 6	2 translations 1 Period 2 rotations 1 Period 3 rotation 1 Period 6 rotation
P6M	The reflections in the three sides of a triangle with angles of 30°, 60° and 90°	Symmetry axes Perpendicular symmetry axes Order 6 turning centre	2 translations 1 Period 2 rotation 1 Period 3 rotation 1 Period 6 rotation 2 glide-reflections 2 reflections

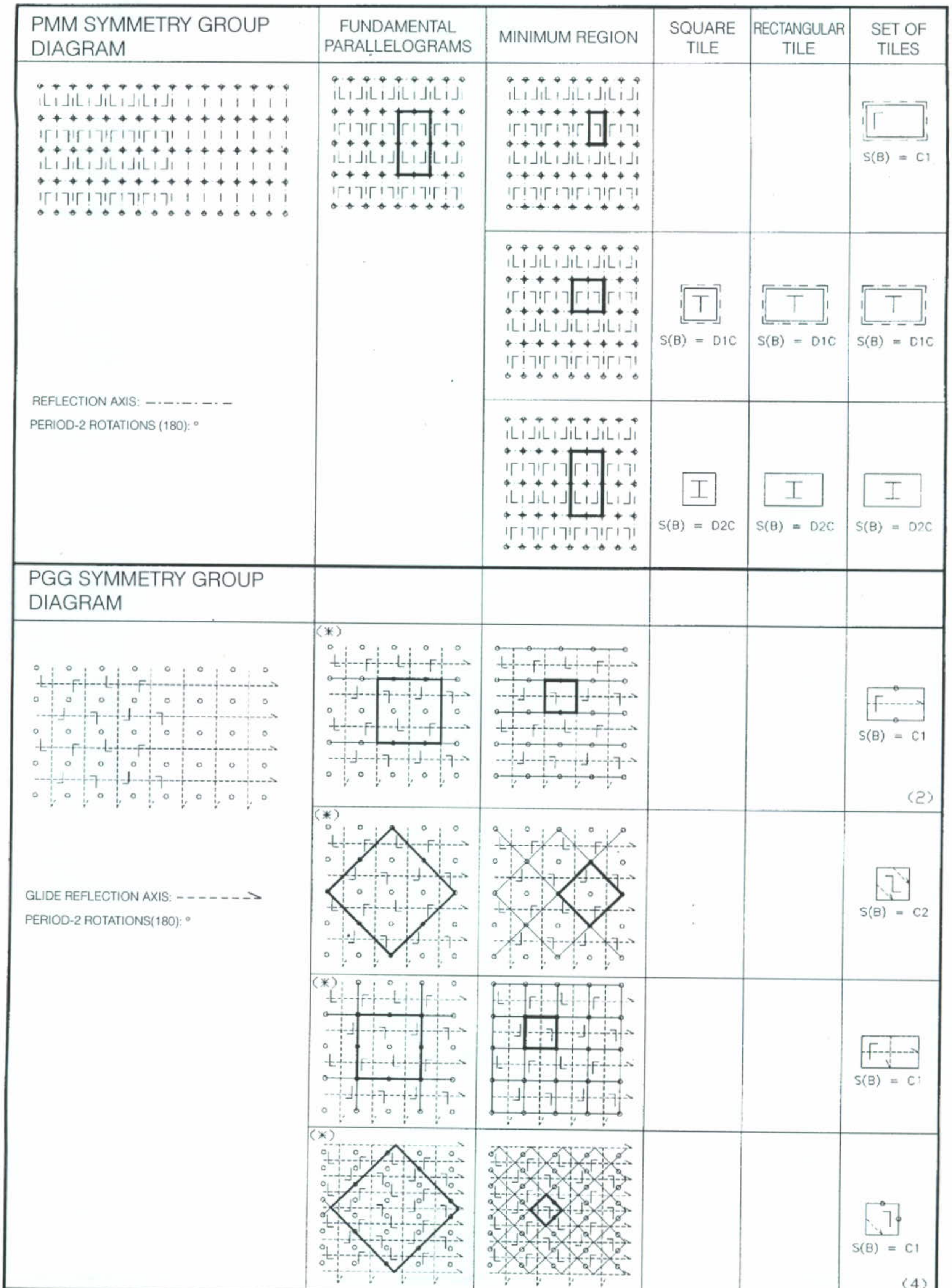
- TABLE 1 -


P1 SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
S(B) = C1
S(B) = C1
S(B) = C1
CM SYMMETRY GROUP DIAGRAM
(*)
S(B) = C1
(2)
S(B) = D1L
S(B) = D1L
REFLECTION AXIS:
GLIDE-REFLECTION AXIS:
PM SYMMETRY GROUP DIAGRAM
S(B) = C1
(2)
S(B) = D1C
S(B) = D1C
S(B) = D1C
REFLECTION AXIS:
PG SYMMETRY GROUP DIAGRAM
(*)
S(B) = C1
(2)
(*)
S(B) = C1
(2)
GLIDE-REFLECTION AXIS:

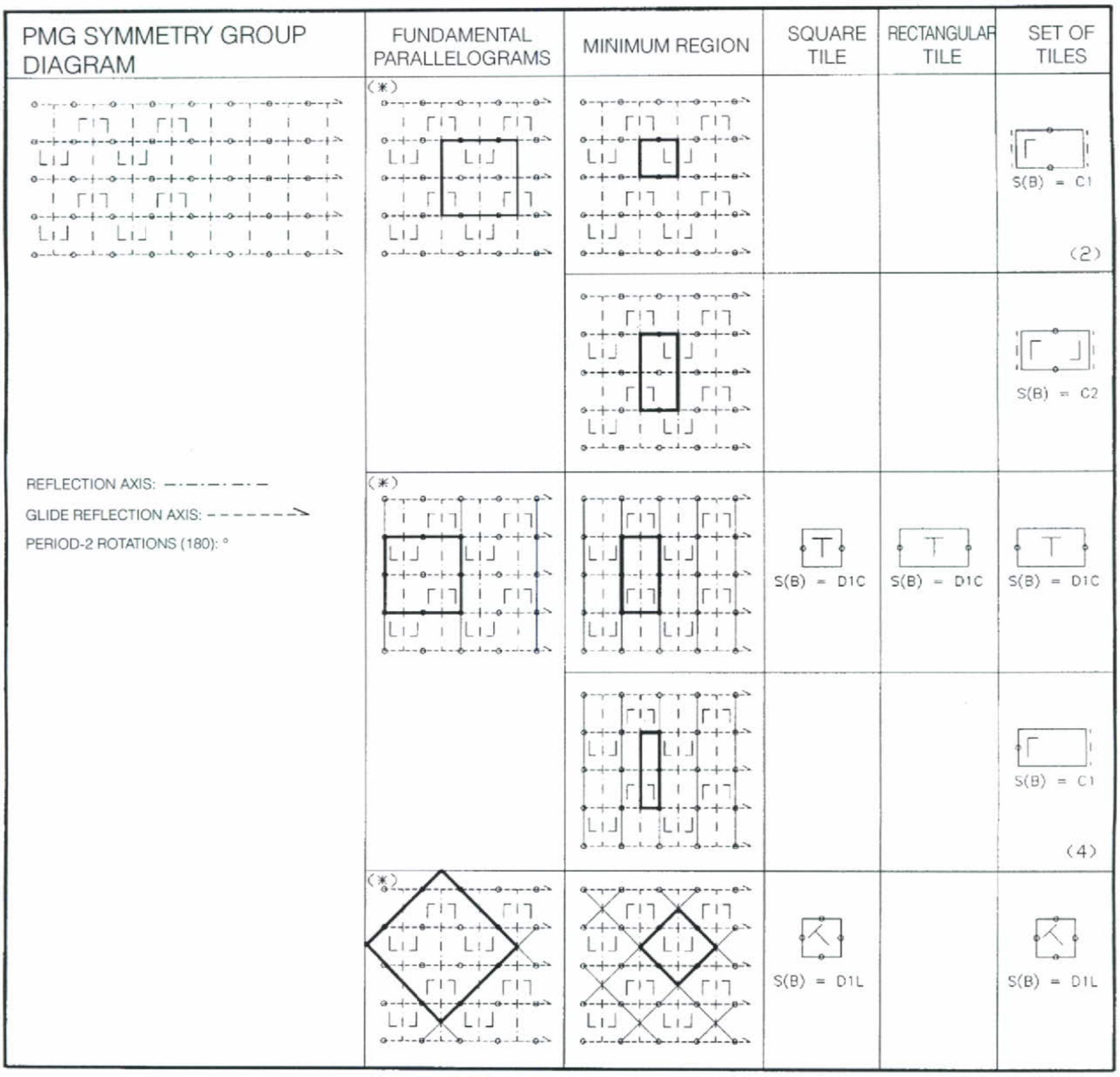
- FIGURE 2 -


P2 SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
(*)
S(B) = C1
S(B) = C1
S(B) = C1
S(B) = C1
S(B) = C1
S(B) = C1
(2)
(2)
(2)
PERIOD-2 ROTATIONS (180): °
S(B) = C2
S(B) = C2
S(B) = C2
CMM SYMMETRY GROUP DIAGRAM
(*)
S(B) = C1
(4)
S(B) = C2
REFLECTION AXIS:
GLIDE REFLECTION AXIS:
PERIOD-2 ROTATIONS(180): °
(*)
S(B) = D1C
S(B) = D1C
S(B) = D1C
S(B) = D2L
S(B) = D2L

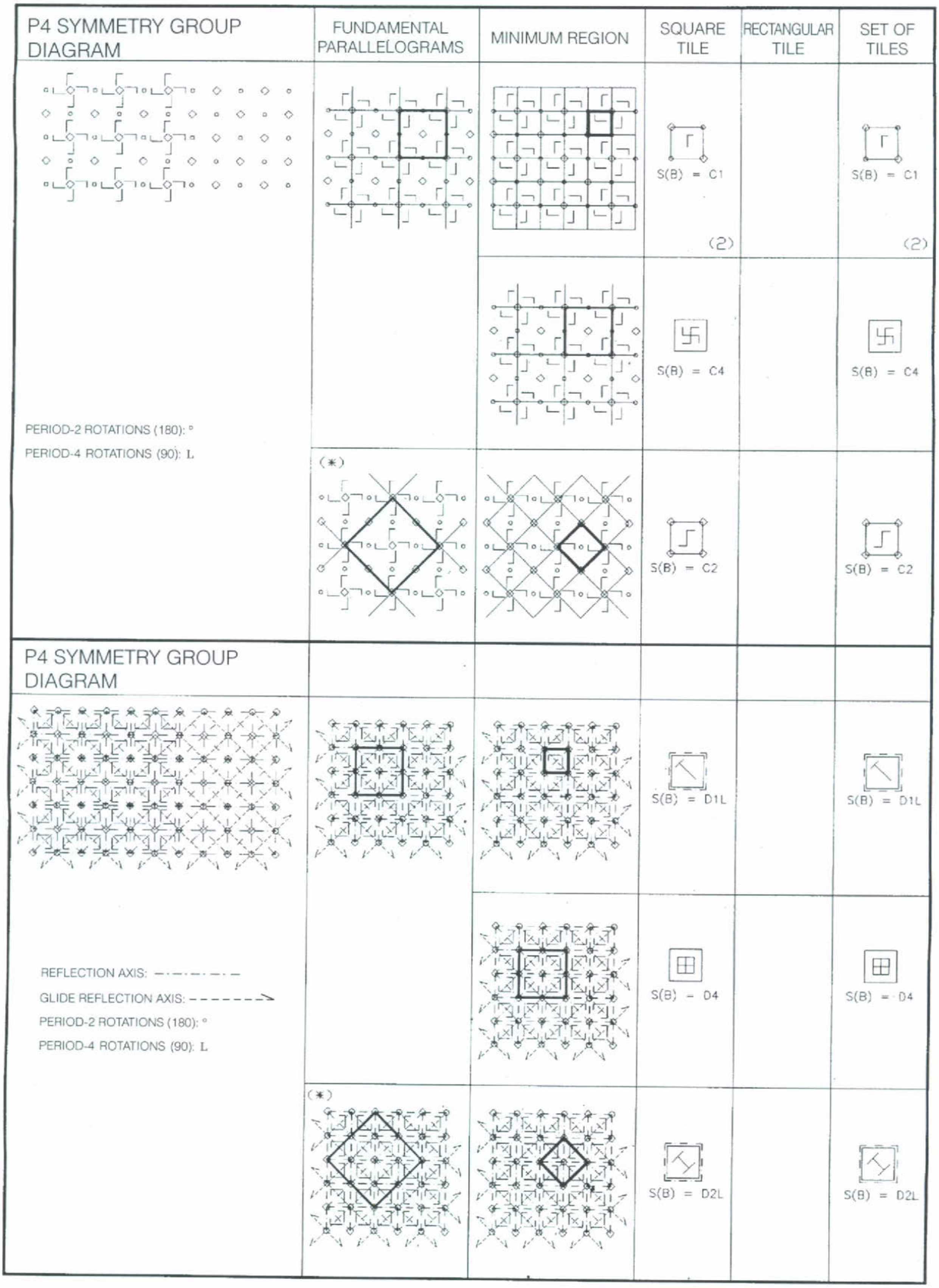
- FIGURE 3 -


PMM SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
S(B) = C1
S(B) = D1C
S(B) = D1C
S(B) = D1C
REFLECTION AXIS:
PERIOD-2 ROTATIONS (180): °
S(B) = D2C
S(B) = D2C
S(B) = D2C
PGG SYMMETRY GROUP DIAGRAM
(*)
S(B) = C1
(2)
GLIDE REFLECTION AXIS:
PERIOD-2 ROTATIONS(180): °
(*)
S(B) = C2
(*)
S(B) = C1
(*)
S(B) = C1
(4)

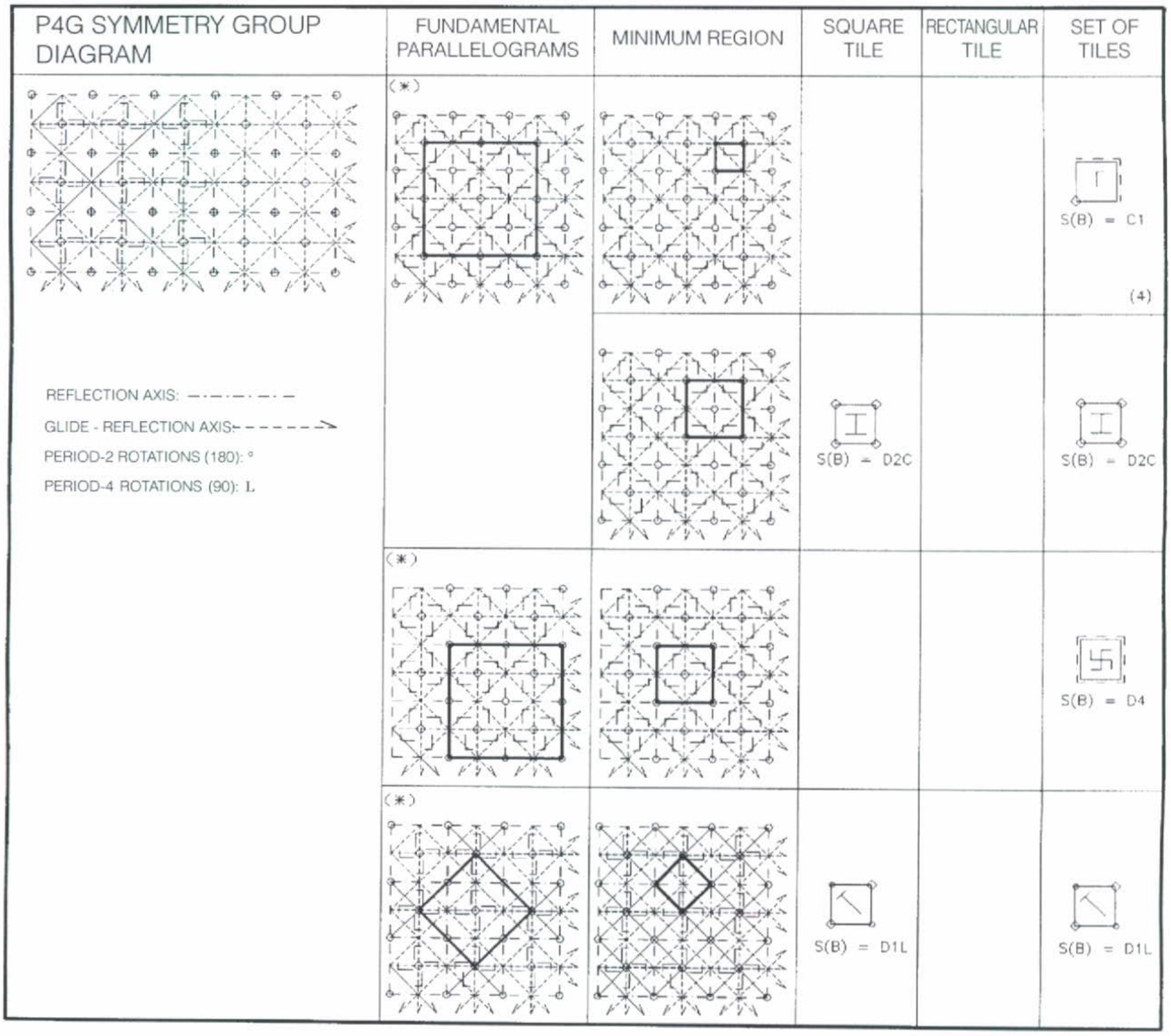
- FIGURE 4 -


PMG SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
REFLECTION AXIS:
GLIDE REFLECTION AXIS:
PERIOD-2 ROTATIONS (180): °
(*)
S(B) = C1
(2)
S(B) = C2
(*)
S(B) = D1C
S(B) = D1C
S(B) = D1C
S(B) = C1
(4)
(*)
S(B) = D1L
S(B) = D1L

- FIGURE 5 -


P4 SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
S(B) = C1
(2)
S(B) = C1
(2)
S(B) = C4
S(B) = C4
PERIOD-2 ROTATIONS (180): °
PERIOD-4 ROTATIONS (90): L
(*)
S(B) = C2
S(B) = C2
P4 SYMMETRY GROUP DIAGRAM
S(B) = D1L
S(B) = D1L
S(B) = D4
S(B) = D4
REFLECTION AXIS: —·—·—·—·—
GLIDE REFLECTION AXIS: - - - - - - ->
PERIOD-2 ROTATIONS (180): °
PERIOD-4 ROTATIONS (90): L
(*)
S(B) = D2L
S(B) = D2L

- FIGURE 6 -


P4G SYMMETRY GROUP DIAGRAM
FUNDAMENTAL PARALLELOGRAMS
MINIMUM REGION
SQUARE TILE
RECTANGULAR TILE
SET OF TILES
(*)
S(B) = C1
(4)
REFLECTION AXIS: —·—·—·—·—
GLIDE - REFLECTION AXIS: - - - - - - ->
PERIOD-2 ROTATIONS (180): °
PERIOD-4 ROTATIONS (90): L
S(B) = D2C
S(B) = D2C
(*)
S(B) = D4
(*)
S(B) = D1L
S(B) = D1L

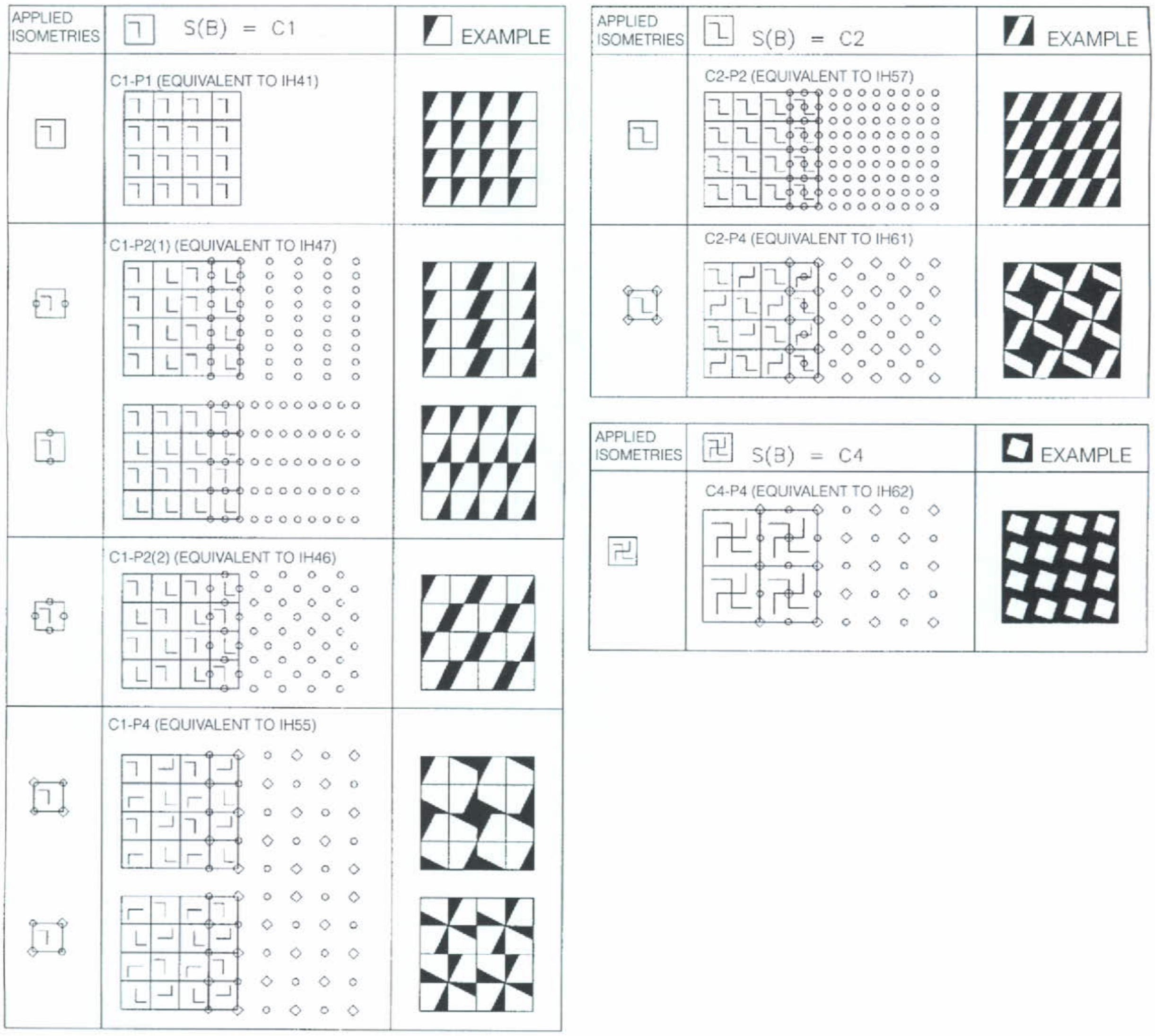
- FIGURE 7 -

SQUARE TILE		RECTANGULAR TILE		SET OF TILES		ISOHEDRAL EQUIVALENT
APPLIED ISOMETRIES	TILING NOMENCLATURE	APPLIED ISOMETRIES	TILING NOMENCLATURE	APPLIED ISOMETRIES	TILING NOMENCLATURE	
S(B)=C1						
	C1-P1		C1,R-P1		C1(X,Y)-P1	IH41
	C1-P2(1)		C1,R-P2(1)		C1(X,Y)-P2(1)	IH47
	C1-P2(2)		C1,R-P2(2)		C1(X,Y)-P2(2)	IH46
	C1-P4				C1(X)-P4	IH55
					C1(X)-PG(1)	IH44
					C1(X,Y)-PG(2)	IH43
					C1(X,Y)-PM	IH42
					C1(X,Y)-CM	IH45
					C1(X,Y)-PMG(1)	IH49
					C1(X,Y)-PMG(2)	IH50
					C1(X,Y)-PMM	IH48
					C1(X,Y)-PGG(1)	IH51
					C1(X,Y)-PGG(2)	IH52
					C1(X)-PGG(3)	IH53
					C1(X,Y)-CMM	IH54
					C1(X)-P4G	IH56
S(B)=C2						
	C2-P2		C2,R-P2		C2(X,Y)-P2	IH57
	C2-P4				C2(X)-P4	IH61
					C2(X,Y)-PMG	IH58
					C2(X)-PGG	IH59
					C2(X,Y)-CMM	IH60
S(B)=C4						
	C4-P4				C4(X)-P4	IH62
					C4(X)-P4G	IH63

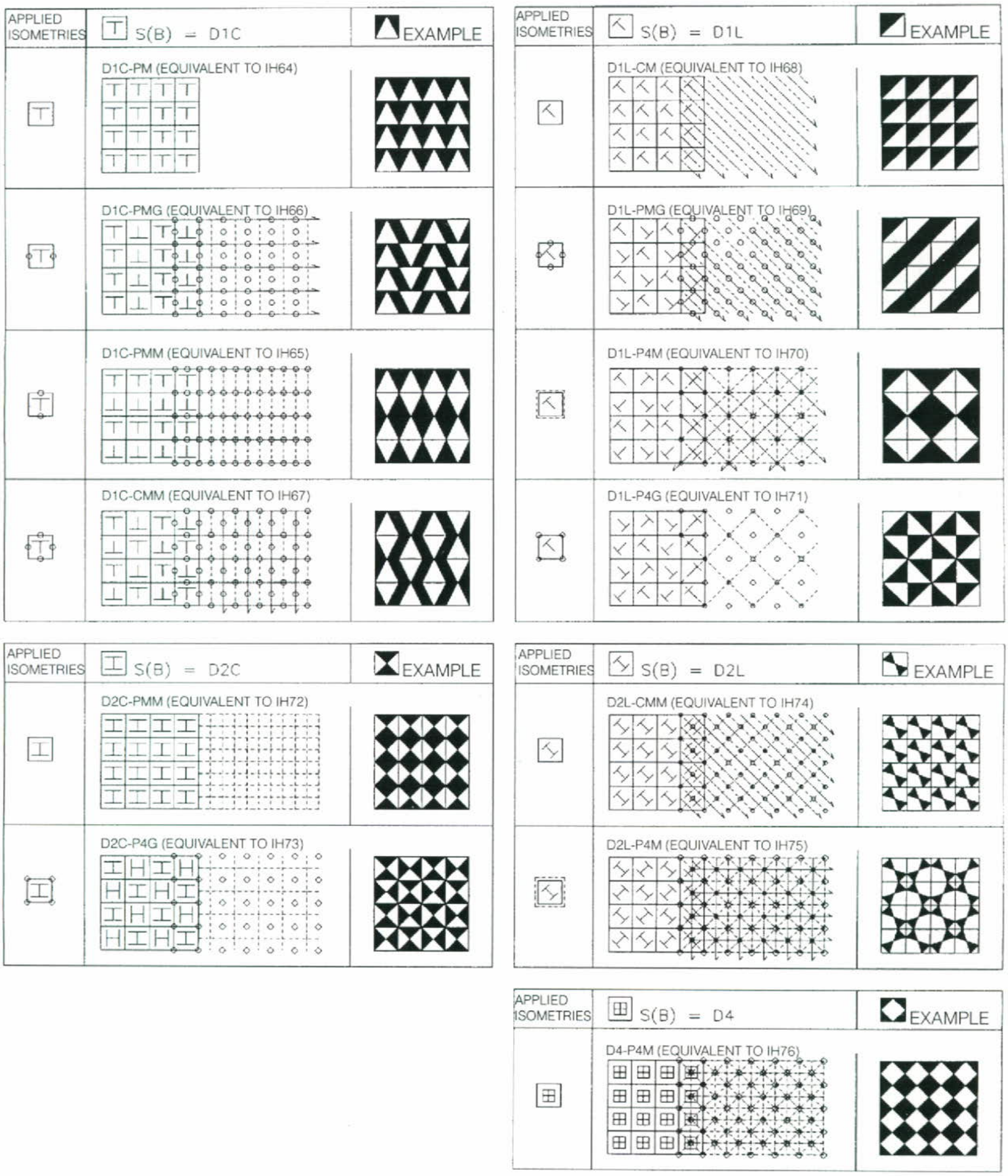
- TABLE 2 -

S(B)=D1C						
SQUARE TILE		RECTANGULAR TILE		SET OF TILES		ISOHEDRAL EQUIVALENT
APPLIED ISOMETRIES	TILING NOMENCLATURE	APPLIED ISOMETRIES	TILING NOMENCLATURE	APPLIED ISOMETRIES	TILING NOMENCLATURE	
	D1C-PM		D1C,R-PM		D1C(X,Y)-PM	IH64
	D1C-PMG		D1C,R-PMG		D1C(X,Y)-PMG	IH66
	D1C-PMM		D1C,R-PMM		D1C(X,Y)-PMM	IH65
	D1C-CMM		D1C,R-CMM		D1C(X,Y)-CMM	IH67
S(B)=D1L						
	D1L-CM				D1L(X)-CM	IH68
	D1L-PMG				D1L(X)-PMG	IH69
	D1L-P4G				D1L(X)-P4G	IH71
	D1L-P4M				D1L(X)-P4M	IH70
S(B)=D2C						
	D2C-PMM		D2C,R-PMM		D2C(X,Y)-PMM	IH72
	D2C-P4G				D2C(X)-P4G	IH73
S(B)=D2L						
	D2L-CMM				D2L(X)-CMM	IH74
	D2L-P4M				D2L(X)-P4M	IH75
S(B)=D4						
	D4-P4M				D4(X)-P4M	IH76

- TABLE 3 -


APPLIED ISOMETRIES
S(B) = C1
EXAMPLE
C1-P1 (EQUIVALENT TO IH41)
C1-P2(1) (EQUIVALENT TO IH47)
C1-P2(2) (EQUIVALENT TO IH46)
C1-P4 (EQUIVALENT TO IH55)
APPLIED ISOMETRIES
S(B) = C2
EXAMPLE
C2-P2 (EQUIVALENT TO IH57)
C2-P4 (EQUIVALENT TO IH61)
APPLIED ISOMETRIES
S(B) = C4
EXAMPLE
C4-P4 (EQUIVALENT TO IH62)

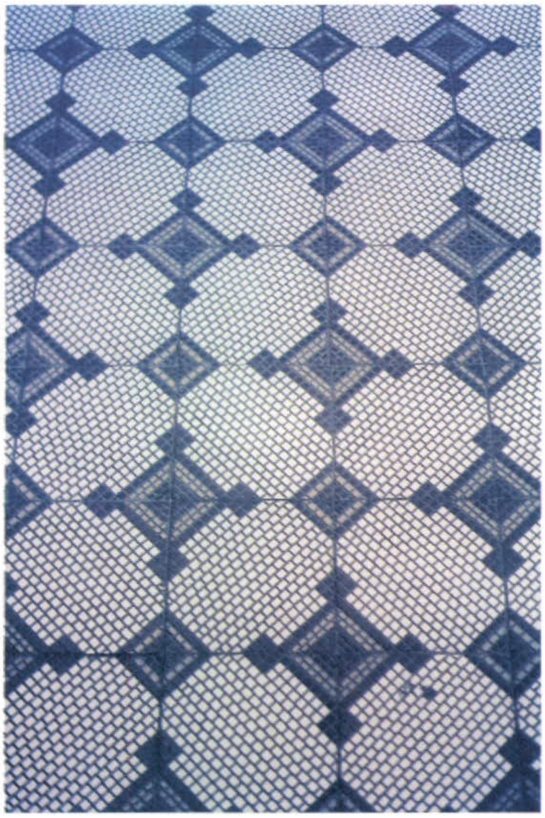
- FIGURE 8 -


APPLIED ISOMETRIES
S(B) = D1C
EXAMPLE
D1C-PM (EQUIVALENT TO IH64)
D1C-PMG (EQUIVALENT TO IH66)
D1C-PMM (EQUIVALENT TO IH65)
D1C-CMM (EQUIVALENT TO IH67)
APPLIED ISOMETRIES
S(B) = D1L
EXAMPLE
D1L-CM (EQUIVALENT TO IH68)
D1L-PMG (EQUIVALENT TO IH69)
D1L-P4M (EQUIVALENT TO IH70)
D1L-P4G (EQUIVALENT TO IH71)
APPLIED ISOMETRIES
S(B) = D2C
EXAMPLE
D2C-PMM (EQUIVALENT TO IH72)
D2C-P4G (EQUIVALENT TO IH73)
APPLIED ISOMETRIES
S(B) = D2L
EXAMPLE
D2L-CMM (EQUIVALENT TO IH74)
D2L-P4M (EQUIVALENT TO IH75)
APPLIED ISOMETRIES
S(B) = D4
EXAMPLE
D4-P4M (EQUIVALENT TO IH76)

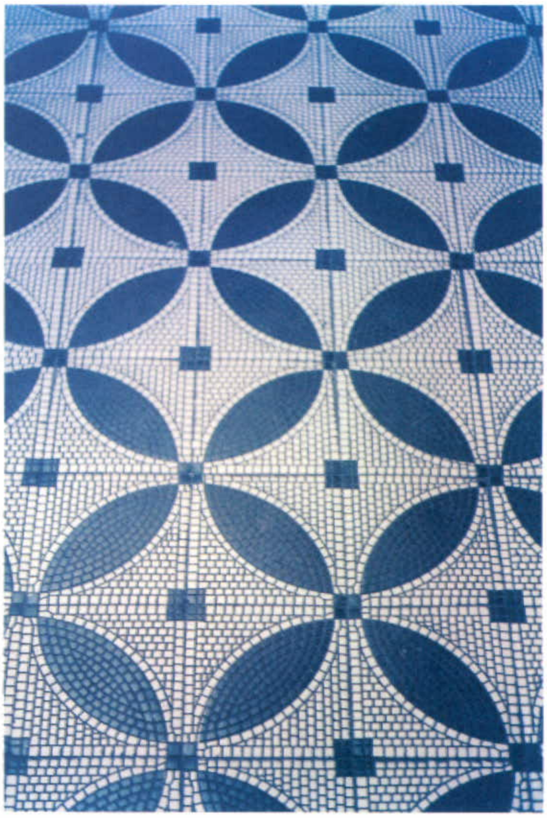
- FIGURE 9 -

		TILE SYMMETRY GROUP								TILING SYMMETRY GROUP											
TILINGS	APPLI-CATIONS	C1	C2	C4	DIC	DIL	D2C	D2L	D4	P1	PM	PG	CM	P2	CMM	PMM	PGG	PMG	P4	P4G	P4M
C1-P1	1	1								1											
C1-P2(1)	0	0												0							
C1-P2(2)	0	0												0							
C1-P4	1	1																	1		
C2-P2	0		0											0							
C2-P4	0		0																0		
C4-P4	8			8															8		
DIC-PM	8				8						8										
DIC-PMG	0				0													0			
DIC-PMM	9				9												9				
DIC-CMM	0				0										0						
DIL-CM	1					1							1								
DIL-PMG	1					1												1			
DIL-P4G	0					0														0	
DIL-P4M	46					46															46
D2C-PMM	7						7									7					
D2C-P4G	0						0													0	
D2L-CMM	1							1							1						
D2L-P4M	7							7													7
D4-P4M	31								31												31
TOTAL	**121**	**2**	**0**	**8**	**17**	**48**	**7**	**8**	**31**	**1**	**8**	**0**	**1**	**0**	**1**	**16**	**0**	**1**	**9**	**0**	**84**

- TABLE 4 -



- FIGURE 10 -



- FIGURE 11 -

- FIGURE 12 -